

Generalized Migdal-Kadanoff Bond-moving Renormalization Recursion Procedure II: Symmetrical Half-length Bond Operation on Fractals

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Abstract

In this second part of the series of two papers we report another type of generalized Migdal-Kadanoff bond-moving renormalization group transformation recursion procedures considering symmetrical single bond operations on fractals. The critical behavior of the spin-continuous Gaussian model constructed on the Sierpinski gaskets is studied as an example to reveal its predominance in application. Results obtained by this means are found to be in good conformity with those obtained from other studies.

PACS numbers: 05.50.+q, 64.60.-i, 75.10.Hk

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I. INTRODUCTION

As has been discussed in the first part of this series of two papers (henceforth referred to as paper I), the Migdal-Kadanoff bond-moving renormalization group transformation recursion procedure and its extensions are very powerful for the study of near-critical properties of various classical lattice systems with globally symmetries [1–3]. However inconveniences can still be found in treating some dual symmetrical systems such as the fractals even if the generalized procedures presented in paper I are used [4–7].

From another point of view, the fractal systems with local symmetries have attracted much attention in the study of phase transition and critical phenomena since the pioneering works by Gefen and co-workers [8–10]. In the past few decades great effort has been devoted to the investigation of such typical fractal systems as Koch-type curves [11], diamond-type hierarchical lattices [12–15], Bethe-type lattices [16] as well as Sierpinski carpets and gaskets [17, 18]. The method of decimation [19, 20], block transformation [21–26] and cumulate expansion [27] are the usual means for scientists to rely on. Yet the applying of bond-moving procedures on these fractals systems is seldom reported.

We noticed the generalized bond-moving transformation recursion procedures presented in paper I can be easily applied to the Sierpinski gaskets under a simple alteration. Thus resulted in this paper another type of generalization of the remarkable Migdal-Kadanoff bond-moving renormalization group transformation. In the following sections of this paper we will give in detail the generalizations we have made on these procedures (Sec. II) and their predominance in applications (Sec. III) respectively by recurring them on the Sierpinski gaskets to investigate the critical properties of the spin-continuous Gaussian model. A summary of our conclusion and some further discussions are presented in Sec. IV.

II. GENERALIZATION

Sierpinski gaskets are a kind of typical infinitely ramified regular fractal lattices that can be easily constructed by a repeated process. The starting point is usually based on a regular triangular as is illustrated in Fig. 1a. Then connecting each midpoint of the three sides to divide the initial triangular to B^2 ($B = 2$ here in this paper) smaller sub-triangles, out of which l^2 ($l = 1$) sub-triangles at the center of the initial triangular is eliminated (Fig.

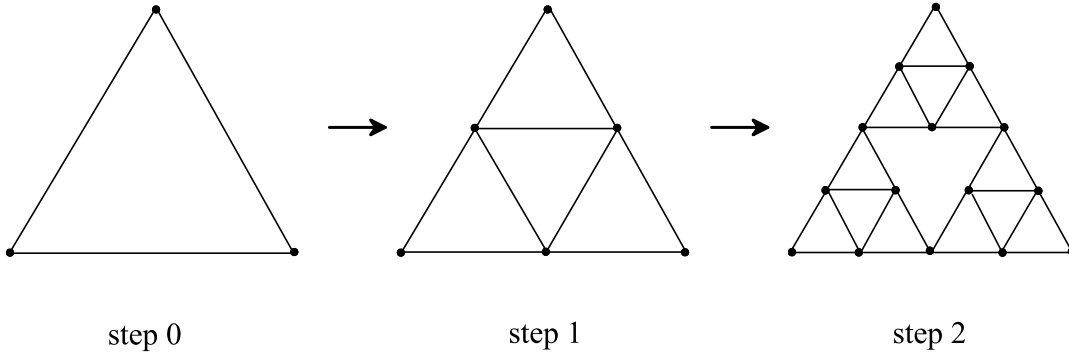


FIG. 1: Recursion procedures for the construction of Sierpinski gaskets.

1b). This procedure is then infinitely repeated in the remained smaller sub-triangles (Fig. 1c) iterating to the microscopic length scales. The fractal dimension of this Sierpinski gaskets is then determined by

$$D_f = \ln(B^2 - l^2)/\ln B = \ln 3/\ln 2. \quad (1)$$

Here we can found that the partial structure of the Sierpinski gaskets is actually still a triangular. Then we can deduce reversely from this fact that it will be very easy to make the Sierpinski gaskets coarse-grained if one proceeds following a very similar renormalization procedure as those we have presented on the triangular lattices in paper I.

Basing on these considerations, we present here in the second part of this series of two papers another type of generalized bond-moving recursion procedures that can be used very conveniently on the Sierpinski gaskets like fractal lattices. It proceeds in such a little different way: (1) selecting a cluster of six lattice sites in a small sub-triangular part of the Sierpinski gaskets as a basic unit for recursion; (2) moving the to be eliminated bonds connecting the three to be eliminated sites in the selected triplet with a weight of half length bonds to the peripheral bonds. For an example bond $2 - 3$ in $\triangle ABC$ connecting sites 2 and 3 is moved with a half length weight to bonds $A - 1$ and $1 - B$ respectively as is shown in Fig.2b; (3) rescaling the system and decimating the to be eliminated sites. Thus the lattice is made coarse-grained under enough steps of renormalization. This procedure can also be proved to be a powerful way bringing with great convenience in particular in the study of

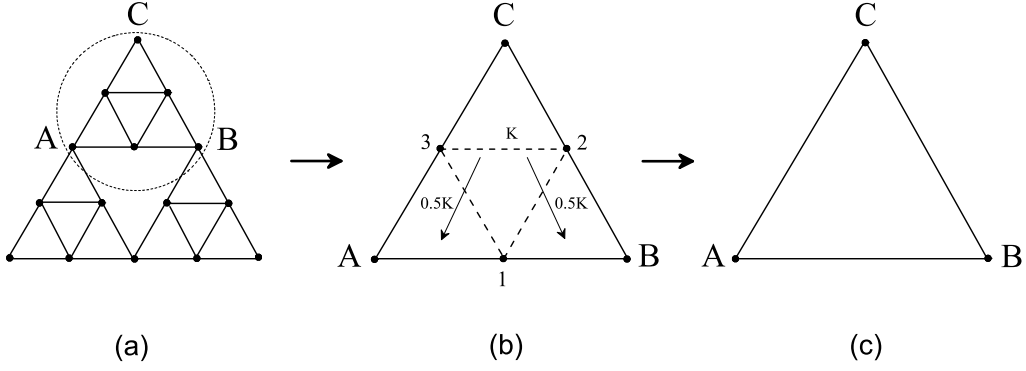


FIG. 2: Renormalization recursion procedures for the Sierpinski gaskets to make the lattice coarse-grained in which symmetrical half-length bond operations are considered.

spin-continuous systems constructed on the fractal-like lattices.

III. PREDOMINANCE ILLUSTRATION

For comparing and predominance illustrating of the above bond-moving recursion procedures with other means, we study again the critical behavior of the spin-continuous Gaussian model constructed on the Sierpinski gaskets whose spins can take any real value between $(-\infty, +\infty)$. The probability of finding a given spin between σ_i and $\sigma_i + d\sigma_i$ is assumed to be $p(\sigma_i)d\sigma_i \propto [\exp -(b/2)\sigma_i^2]d\sigma_i$. This results in the classical Gaussian effective Hamiltonian with two-body nearest-neighbor interactions

$$H_{\text{eff}} = \sum_{\langle ij \rangle} K \sigma_i \sigma_j - \frac{b}{2} \sum_i \sigma_i^2, \quad (2)$$

where $K = J/k_B T$ is the reduced nearest-neighbor interaction with $K > 0$ denotes the ferromagnetic systems; b is the Gaussian distribution constant; k_B the Boltzmann constant and T the thermodynamic temperature. The summation of the spin is performed between each nearest-neighbor pair $\langle ij \rangle$.

In order to successfully complete the bond-moving and decimation processes and generalizing the Gaussian model on translational invariant lattices to that on the Sierpinski gaskets, we assign two types of interactions K_e and K for differentiation of spin interaction at different cases as Gefen et al did in previous studies [10]. Where the bond noted as K_e

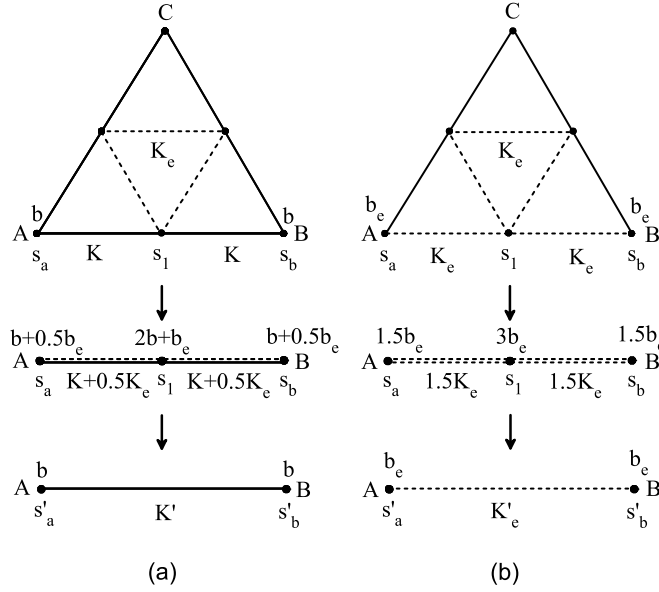


FIG. 3: Bond-moving and decimation processes to generalizing the Gaussian model on translational invariant lattices to that on the Sierpinski gaskets.

separates between two to be eliminated sub-triangles while K borders a non-eliminated one. For the particular case of Gaussian model two types of self-energy $(-b_e s^2/2)$ and $(-b s^2/2)$ should also be assigned correspondingly but the numerical value of b_e and b is actually identical as well as that of K_e and K . Thus if a spin has N bonds of K and N_e bonds of K_e with its nearest-neighbors, the self-energy of it is then given by $[-(N_e b_e + N b) s^2/2]$. In the renormalization procedures the to be eliminated bonds are moved with a weight of half length to the peripheral ones resulting a half weight addition to the nearest-neighbor spin interaction but a wholly symmetrical maintaining of the lattice.

As is illustrated in Fig.3, along with the moving of the nearest-neighbor interaction $K s_i s_j$ (or $K_e s_i s_j$) between the spins s_i and s_j , the self-energy $[-b(s_i^2 + s_j^2)/2]$ (or correspondingly $[-b_e(s_i^2 + s_j^2)/2]$) move in the same direction. The decimation procedure for the renormalized bond K' is

$$\begin{aligned}
 & \int_{-\infty}^{+\infty} \exp \left[(K + 0.5K_e) (s_a s_1 + s_1 s_b) - \frac{b + 0.5b_e}{2} (s_a^2 + s_b^2) - \frac{2b + b_e}{2} s_1^2 \right] ds_1 \\
 & = C \exp \left[K' s'_a s'_b - \frac{b}{2} (s_a'^2 + s_b'^2) \right].
 \end{aligned} \tag{3}$$

By directly integrating s_1 to decimate the intermediate spins it becomes

$$\begin{aligned} C \exp \left[\left(\frac{(2K + K_e)^2}{4(2b + b_e)} \right) s_a s_b + \left(\frac{(2K + K_e)^2}{16b + 8b_e} - \frac{2b + b_e}{4} \right) (s_a^2 + s_b^2) \right] \\ = C \exp \left[K' s'_a s'_b - \frac{b}{2} (s'^2_a + s'^2_b) \right]. \end{aligned} \quad (4)$$

For the continuity of spin sampling, the spins are rescaled by

$$s'_a = \xi_a s_a \quad \text{and} \quad s'_b = \xi_b s_b \quad (5)$$

with

$$\xi_a^2 = \xi_b^2 = 1 + \frac{b_e}{2b} - \frac{(2K + K_e)^2}{4b(2b + b_e)}. \quad (6)$$

Then the recursion relation for K' is obtained to be

$$K' = R(K, K_e) = \frac{b(2K + K_e)^2}{2(2b + b_e)^2 - (2K + K_e)^2}. \quad (7)$$

Meanwhile, The renormalization group transformation of the cell decimated to the renormalized bond K'_e is

$$\begin{aligned} \int_{-\infty}^{+\infty} \exp \left[\frac{3}{2} K_e (s_a s_1 + s_1 s_b) - \frac{3b_e}{4} (s_a^2 + s_b^2) - \frac{3b_e}{2} s_1^2 \right] ds_1 \\ = C \exp \left[\frac{3K_e^2}{4b_e} s_a s_b - \frac{3(2b_e^2 - K_e^2)}{8b_e} (s_a^2 + s_b^2) \right] \\ = C \exp \left[K'_e s'_a s'_b - \frac{b_e}{2} (s'^2_a + s'^2_b) \right], \end{aligned} \quad (8)$$

from which the recursion relation for K'_e is derived to be

$$K'_e = R_e(K, K_e) = \frac{b_e K_e^2}{2b_e^2 - K_e^2}. \quad (9)$$

From these recursion relations we can found the critical behavior of the Gaussian model on the Sierpinski gaskets is quite different from that on the translational invariant lattices. Here we can obtain an attractive fixed point ($K^* = 0, K_e^* = 0$) and two critical points ($K^* = b, K_e^* = 0$) and ($K^* = 0, K_e^* = b_e$). An fixed point ($K^* = b, K_e^* = b_e$) was found to be repulsive corresponding to the critical point of the Gaussian model on the translational invariant lattices given $K_e = K$ and $b_e = b$.

Since the integration of the partition function are kept limited because of the introduction of self-energies, we can obtain the renormalization-group transformation matrix at ($K^* =$

$b, K_e^* = 0$) as

$$\begin{aligned} R(K, K_e) &= \begin{pmatrix} \frac{\partial K'}{\partial K} & \frac{\partial K'}{\partial K_e} \\ \frac{\partial K'_e}{\partial K} & \frac{\partial K'_e}{\partial K_e} \end{pmatrix}_{(K^*=b, K_e^*=0)} \\ &= \begin{pmatrix} 4 & 2 \\ 0 & 0 \end{pmatrix}, \end{aligned} \quad (10)$$

by setting $b_e = 0$ at critical point ($K^* = b, K_e^* = 0$). Clearly we can found that it has only two eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 0$. Thus we obtain the critical exponent of correlation length as

$$\nu = \frac{\ln B}{\ln \lambda_1} = \frac{\ln 2}{\ln 4} = 0.5, \quad (11)$$

in good conformity with the previous results [17, 28, 29].

In the meantime, the renormalization-group transformation matrix at ($K^* = 0, K_e^* = b_e$) is found to be

$$\begin{aligned} R_e(K, K_e) &= \begin{pmatrix} \frac{\partial K'}{\partial K} & \frac{\partial K'}{\partial K_e} \\ \frac{\partial K'_e}{\partial K} & \frac{\partial K'_e}{\partial K_e} \end{pmatrix}_{(K^*=0, K_e^*=b_e)} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}, \end{aligned} \quad (12)$$

by setting $b = 0$. It has also two eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 4$ resulting in the correlation length critical exponent

$$\nu_e = \frac{\ln B}{\ln \lambda_2} = \frac{\ln 2}{\ln 4} = 0.5 \quad (13)$$

conforming to the previous results.

IV. SUMMARY AND DISCUSSION

In summary, in this second part of the series of two papers we have generalized the Migdal-Kadanoff bond-moving renormalization group transformation recursion procedures to containing symmetrical single bond operations that can be used conveniently in particular on the fractal lattices. The critical behavior of the classical spin-continuous Gaussian model constructed on the Sierpinski gaskets was studied as an example of the application of these

procedures. Results obtained are in good conformity with previous studies revealing the dependability of these means. The predominance of these procedures revealed in this paper may encourage many future applications of them on some more complicated spin systems such as the S^4 models.

ACKNOWLEDGEMENTS

This work was supported by the Shandong Province Science Foundation for Youths (Grant No.ZR2011AQ016), the Shandong Province Postdoctoral Innovation Program Foundation (Grant No.201002015), the Scientific Research Starting Foundation, Youth Foundation (Grant No.XJ201009) and the Foundation of Scientific Research Training Plan for Undergraduate Students (Grant No.2010A023) of Qufu Normal University.

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